

On Number of Compositions of Natural Numbers

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Abstract

We first give a combinatorial interpretation of coefficients of Chebyshev polynomials, which allows to connect them with compositions of natural numbers. Then we describe a relationship between the number of compositions of a natural number in which a certain number of parts are $p-1$, and other parts are $\geq p$, with compositions in which all parts are $\geq p$. Then we find a relationship between principal minors of a type of Hessenberg matrices and compositions of natural numbers.

1 Introduction

We first prove some combinatorial results which will be used in the paper. In the second section we prove a formula for some restricted compositions of natural numbers.

In the second section we show that coefficients of Chebyshev polynomials may be interpreted in terms of compositions of natural numbers.

Next, in the third section, we connect compositions in which some parts are $p-1$, and all other parts $\geq p$ with compositions in which all parts are $\geq p$.

Different mathematical objects may be represented by determinants of Hessenberg matrices. We describe a special kind of Hessenberg matrices which determinants are compositions of natural numbers. It will also be

shown that sums of principal minors of these matrices, that is, coefficients of its characteristic polynomials also represent some type of compositions of natural numbers. This is done in the fourth section.

We shall use the following result about Hessenberg's matrices, which is easily proved by induction.

Theorem 1. *Let $a_1, p_{i,j}$, ($i \leq j$) be arbitrary elements of a commutative ring R , and let the sequence a_1, a_2, \dots be defined by*

$$a_{n+1} = \sum_{i=1}^n p_{i,n} a_i, \quad (n = 1, 2, \dots),$$

If

$$P_n = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ 0 & 0 & 0 & \cdots & -1 & p_{n,n} \end{bmatrix},$$

then

$$a_{n+1} = a_1 \det A_n, \quad (n = 1, 2, \dots).$$

We prove now a combinatorial result which will also be used in the paper.

Consider a set X consisting of n blocks X_i each having 2 elements, and an additional block Y with m elements. We call X_i the main blocks of X . By an $n+k$ -inset of X we shall mean a subset of X with $n+k$ elements intersecting each main block X_i , ($i = 1, 2, \dots, n$). We shall denote by $N(n, k, m)$ the number of $n+k$ -insets of X . For this number we have

Proposition 1. *Let n be a positive integer, and let k, m be nonnegative integers. Then*

$$N(n, k, m) = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{2n-2i+m}{n+k}.$$

Also,

$$N(n, k, m) = 2^{n-k} \sum_{i=0}^m 2^i \binom{m}{i} \binom{n}{k-i}.$$

Proof. For $i = 1, 2, \dots, n$ and an $n+k$ -subset Z of X define the property i to be:

Z does not intersect X_i .

By inclusion-exclusion principle we obtain

$$f(n, k, m) = \sum_{I \subseteq [n]} (-1)^{|I|} N(I),$$

where $N(I)$ is the number of $n + k$ -subsets of X which do not intersect main blocks X_i , ($i \in I$). There are

$$\binom{n}{i} \binom{2n + m - 2i}{n + k}$$

such subsets and the first formula is proved.

We may count $n + k$ -insets of X in another way. We prove firstly the following formula:

$$N(n, k, m) = \sum_{i=0}^m \binom{m}{i} N(n, k - i, 0). \quad (1)$$

In fact, omitting the additional block Y we obtain the set X_1 with no additional block.

Each $n + k$ -inset of X_1 is an $n + k$ -inset of X . There are $N(n, k, 0)$ such insets. In this way we obtain all $n + k$ -inset of X not intersecting Y .

The remaining $n + k$ -insets of X are obtained as a union of some $n + k - i$, ($1 \leq i \leq m$)-inset of X_1 and some i -set of additional block Y . This proves (1).

Further, we obtain $N(n, k, 0)$ by choosing two elements from each of arbitrary k main blocks, and one element from each of remaining $n - k$ main blocks. The first may be done in $\binom{n}{k}$ ways and the second in 2^{n-k} ways. We conclude that $N(n, k, 0) = 2^{n-k} \binom{n}{k}$. It follows that

$$N(n, k, m) = 2^{n-k} \sum_{i=0}^m 2^i \binom{m}{i} \binom{n}{k-i},$$

and the second formula is true. □

2 Chebyshev Polynomials and Compositions

In this section we shall show that coefficients of Chebyshev polynomials may be interpreted in terms of compositions of natural numbers.

Let x be a symbol. By an $u(n, k, x)$ -sequence we shall mean a sequence of length n in which k terms are equal x , and remaining terms are either 1 or $1|$. The sequence does not end by $1|$.

Proposition 2. *The number of $u(n, k, x)$ -sequences is $N(n - 1, k, 1)$.*

Proof. We prove that all $u(n, k, x)$ -sequences may be labeled by $n - 1 + k$ -insets of a set X consisting of $n - 1$ main blocks, each with 2 elements, and an additional block Y with one element.

Let Z be an $n - 1 + k$ -inset of X . Form a $u(n, k, x)$ sequence as follows. Put either x at the last place, if the element from Y is in Z , or put 1 if the element from Y is not in Z . Put then the remaining x 's on positions corresponding to blocks from which both elements are in Z . Then insert 1's on the remaining places to obtain a n -sequence in which k elements are x 's, and the remaining $n - k$ elements are 1's. The last element in the sequence is either x or 1.

If the last element is x then we have $n - k$ main blocks from which only one element is in Z . Put these blocks and all 1's into a bijective correspondence. Each 1, is either followed by $|$, if the second element of corresponding block is in Z , or not followed by $|$, if the first element of corresponding block is in Z .

If the sequence ends by 1 then there are $n - 1 - k$ main blocks from which one element is in Z . We now put these block into a bijective correspondence with 1's, excluding the last one. In such way we also obtain a $u(n, k, x)$ -sequence. It follows that each $n - 1 + k$ -inset of X determines a $u(n, k, x)$ -sequence. Clearly, that different insets produce different sequences.

Conversely, consider a $u(n, k, x)$ -sequence. Form a $n - 1 + k$ inset Z of X in the following way: If the last element in the sequence is x then put element of Y in Z . Otherwise this element is not in Z .

Furthermore, put in Z both elements of main blocks which indices correspond to the indices of remaining x in $u(n, k, x)$. The indices of the remaining main blocks correspond to 1's, excluding eventually the last 1. If 1 corresponding to such main block has $|$ then put the second element of the block in Z . Otherwise put the first element in Z . Thus, Z is a $n - 1 + k$ -inset of X . \square

In the paper [2] a family of polynomials $P_n(x)$ is defined such that the

coefficient by x^{n-k+m} of $P_{n+k+m}(x)$ is $(-1)^k N(n, k, m)$. Particulary, for $m = 0$ we have Chebyshev polynomials $U(n, x)$ of the second kind, and for $m = 1$ we obtain Chebyshev polynomials $T_{n,k}(x)$, of the first kind. Using Propositions 1 and 2 we obtain

Proposition 3. *Let $t_{n,k}$ be the coefficient of Chebyshev polynomial $T_{n+k+1}(x)$ by x^{n-k+1} . Then $(-1)^k t_{n,k}$ is equal to the number of $u(n+1, k, x)$ - sequences. Particulary, we have the following two explicit formulas for $t_{n,k}$.*

$$(-1)^k t_{n,k} = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{2n-2i+1}{n+k}.$$

Also,

$$(-1)^k t_{n,k} = 2^{n-k} \left[\binom{n}{k} + 2 \binom{n}{k-1} \right].$$

There is a natural relationship between $u(n, k, x)$ - sequences and particular kind of compositions of natural numbers. We shall give one such relationship. In fact, if a is a nonnegative integer then sequences $s(n, k, a|)$ produce compositions of $n+k(a-1)$, obtained by adding numbers not separated by $|$. But, we want that different sequences give different compositions. This is not the case if $a \leq n-k$, since, for example, $11|2|$ and $2|11$ generate the same composition $2, 2$ of 4 .

Proposition 4. *Let k, n be nonnegative integers such that $k \leq n+1$. Then $(-1)^k t_{n,k}$ is the number of compositions of $(n+1)(k+1) - k^2$ in which exactly k parts are $\geq n-k+1$.*

Proof. Denote $a = n-k+1$. It is clear that each $s(n, k, a|)$ -sequence produces a required composition. Different sequences give different composition. Namely, consider the composition (i_1, \dots, i_m) obtained from two $u(n, k, a|)$ -sequences. Each i_t , ($t = 1, \dots, m$) may be obtained in two ways: either from a term of the form $11 \cdots 1|$ or from a term of the form $11 \cdots 1a|$. Since no two such terms are equal it follows that i_t must be obtain in the same way in both sequences. Hence, the sequences are the same.

Conversely, let (i_1, i_2, \dots, i_m) be a composition of $(n+1)(k+1) - k^2$ in which exactly k parts are $\geq n-k+1$. Replace each such part by $11 \cdots x$, and all other parts by $11 \cdots 1|$ to obtain a $u(n, k, x)$ sequence. The formula now follows from Proposition 3.

In the case $k = 1$, $n = 3$ we have $a = 3$ and the following 8 compositions of 5:

$$(3, 2), (3, 1, 1), (4, 1), (1, 3, 1), (5), (1, 4), (1, 1, 3), (2, 3).$$

3 Some Restricted Compositions

Let n, p be positive integers. We shall denote by $c(n, p)$ the number of compositions of n in which all parts are $\geq p$. It is convenient to define $c(0, p) = 1$. We know that $c(n, 1) = 2^{n-1}$, $c(n, 2) = f_{n-1}$, where f_{n-1} is Fibonacci number.

If, additionally, k is a nonnegative integer then by $c(n, k, p, p-1)$ will be denoted the number of compositions of n in which exactly k parts are equal $p-1$, and all other parts are $\geq p$.

The following result will be a formula which connects $c(n+kp+1, k, p, p-1)$ with $c(n, p)$.

Proposition 4. *Let n, p be positive integers, and let k be a nonnegative integer. Then*

$$c(n+kp+1, k, p, p-1) = \sum_{j_1+j_2+\dots+j_{k+1}=n} c(j_1+1, p) c(j_2+1, p) \cdots c(j_{k+1}+1, p), \quad (2)$$

where the sum is taken over $j_t \geq -1$, ($t = 1, 2, \dots, k+1$).

Proof. We use the induction with respect of k . For $k = 0$, the assertion is obviously true.

Assume that the assertion is true for $k-1$. The greatest value of j_{k+1} is $n+k$, and is obtained for $j_1 = j_2 = \dots = j_k = -1$. Hence, we may write (2) in the form:

$$c(n+nk+1, k, p, p-1) = \sum_{j=-1}^{n+k} c(j+1, p) \sum_{j_1+j_2+\dots+j_k=n-j} c(j_1+1, p) \cdots c(j_k+1, p).$$

By the induction hypothesis we have

$$c(n+kp+1, k, p, p-1) = \sum_{j=-1}^{n+k} c(j+1, p) \cdot c(n+kp+1-j-p, k-1, p, p-1). \quad (3)$$

Let (i_1, i_2, \dots) be a composition of n with exactly k parts equal $p-1$, and all other parts are $\geq p$. Assume that the first $p-1$ is i_m . Then $(i_1, i_2, \dots, i_{m-1})$ is a composition of $j+1 = i_1 + \dots + i_{m-1}$ in which all parts are $\geq p$. Also, (i_{m+1}, \dots) is a composition of $n+kp+1-j-p$ with exactly $k-1$ parts equal $p-1$, and all other parts $\geq p$.

It follows that the number of compositions in which the first $p - 1$ is at the m th place is $c(j + 1, p) \cdot c(n - j - p, k - 1, p, p - 1)$. This is a term in the sum on the right side of (3).

For $m = 1$ we have $j = -1$ which produces the first term in (3). If $p - 1$ are on the last k places of a composition then $j = n + k$ that gives the last term in the sum of (3). Summing over all j we conclude that the proposition is true. \square

As an immediate consequence of Proposition 4, for $p = 1$, we have

Corollary 2. *Let n be a positive integer, and let k be a nonnegative integer. Then the number K of weak compositions of $n + 1$ in which exactly k parts are equal 0 is*

$$K = \sum_{j_1 + j_2 + \dots + j_{k+1} = n} c(j_1 + 1, 1) c(j_2 + 1, 1) \cdots c(j_k + 1, 1),$$

where $j_t \geq -1$, ($t = 1, 2, \dots, k + 1$), and $c(0, 1) = 1$, $c(j, 1) = 2^{j-1}$, ($j > 0$).

Also, for $p = 2$ we have

Corollary 3. *Let n be a positive integer, and let k be a nonnegative integer. Then the number K_1 of composition of $n + 2k + 1$ in which exactly k parts are equal 1, and all other part are ≥ 2 is*

$$K_1 = \sum_{j_1 + j_2 + \dots + j_{k+1} = n} f_{j_1} \cdots f_{j_{k+1}},$$

where $j_t \geq -1$, ($t = 1, 2, \dots, k + 1$), and f_s are Fibonacci numbers.

4 Principal Minors of Some Hessenberg Matrices

For positive integers p, n we define an upper Hessenberg matrix $F_{n,p} = (F_{n,p}(i, j))_{n \times n}$ in the following way:

$$F_{n,p}(i, j) = \begin{cases} -1 & i = j + 1, \\ 1 & j - i \geq p - 1 \\ 0 & \text{otherwise} \end{cases}.$$

By Theorem 1 we easily obtain that

$$\det F_{1,p} = \det F_{2,p} = \det F_{p-1,p} = 0, \det F_{p,p} = 1,$$

and,

$$\det F_{n,p} = \sum_{i=1}^{n-p+1} \det F_{i,p},$$

that is,

$$\det F_{n,p} = \det F_{n-p,p} + \det F_{n-1,p}.$$

According to [1, p.63] we have

Proposition 5. *Let n, p be positive integers. Then $c(n, p) = \det F_{n,p}$.*

Now, we shall find a relationship between coefficients of characteristic polynomials of matrices $F_{n,p}$ and compositions mentioned in the preceding section.

Proposition 6. *Let n, p be positive integers, and let k be a nonnegative integer. Then the number $c(n + kp - 2k, k, p, p - 1)$ is the sum of all principal minors of the order $n - k$ of the matrix $F_{n,p}$.*

Proof. Denote $M(i_1, \dots, i_k)$ the minor of order $n - k$ obtained by deleting rows and columns of $F_{n,p}$ which indices are $1 \leq i_1 < i_2 < \dots < i_k \leq n$. It is easy to see that the following equation holds

$$M(i_1, \dots, i_k) = \det F_{i_1-1,p} \cdot \det F_{i_2-i_1-1,p} \cdots \det F_{i_k-i_{k-1}-1,p} \cdot \det F_{n-i_k,p}.$$

According to Propositions 5 we have

$$M(i_1, \dots, i_k) = c(i_1 - 1, p) \cdot c(i_2 - i_1 - 1, p) \cdots c(n - i_k, p), \quad (4)$$

Denote $i_1 - 1 = j_1 + 1, i_2 - i_1 - 1 = j_2 + 1, \dots, i_k - i_{k-1} - 1 = j_k + 1, n - i_k = j_{k+1} + 1$ to obtain that $j_1 + \dots + j_{k+1} = n - 2k - 1, (j_t \geq -1, (t = 1, 2, \dots, k + 1))$.

It follows that

$$M(i_1, \dots, i_k) = c(j_1 + 1, p) \cdot c(j_2 + 1, p) \cdots c(j_{k+1} + 1, p).$$

Summing over all $1 \leq i_1 < i_2 < \dots < i_k \leq n$ we obtain a sum S such that

$$S = \sum_{j_1 + \dots + j_{k+1} = n - 2k - 1} c(j_1 + 1, p) \cdot c(j_2 + 1, p) \cdots c(j_{k+1} + 1, p),$$

where the sum is taken over all $j_t \geq -1, (t = 1, 2, \dots, k + 1)$.

The proposition follows from Proposition 4. \square

Since coefficients of the characteristic polynomial of a matrix are up to sign equal to the sum of its principal minors we may state Proposition 5 in the following form:

Corollary 4. *If $(-1)^{n-k}a_{n-k}$ is the coefficient by x^k of the characteristic polynomial of the matrix $F_{n,p}$ then a_{n-k} is the number of compositions of $n + kp - 2k$, in which there are k parts equal $p - 1$ and all other parts are $\geq p$.*

We state two particular cases of Proposition 5, for $p = 1$ and $p = 2$.

Corollary 5. *The number of weak compositions of $n - k$, ($k \leq n$) in which exactly k parts are 0, is equal to the sum of all principal minors of order $n - k$ of the matrix $F_{n,1}$.*

Corollary 6. *The number of compositions of n , in which exactly k , ($k \leq n - 2$) parts are 1, and all other parts are ≥ 2 , is equal to the sum of all principal minors of order $n - k$ of the matrix $F_{n,2}$.*

References

- [1] G. E. Andrews *The Theory of Partitions*, Addison-Wesley, 1976.
- [2] M. Janjić, On a Class of Polynomials with Integer Coefficients, *Journal of Integer Sequences*, Volume 11, Issue 5, a:08.5.2, 2008.